Twelve Computational Geometry Problems from Cartographic Generalization

(draft)

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Abstract

The topic of automated cartographic generalization is discussed as a source of open problems for computational geometers. To this end, an introduction in the geometric conditions and operations that occur is given first. Then these conditions and operations are formalized, leading to various different problem statements for the design of algorithms that perform a step in cartographic generalization.

Note: This paper is in preliminary form; it was written for computational geometers rather than GIS researchers, and the list of references must be extended. Figures are also in preliminary form (the html-version doesn't have boldface lines where it should).

A preliminary copy this paper will be distributed during a Dagstuhl Seminar on Computational Geometry (March 1999). Hopefully, this will invoke research on these problems, and the author has requested be informed about any progress and results. In a more polished form, hopefully with references to new results, approaches, ideas, problem statements, it will be submitted to a workshop on automated generalization, where the author would like to report on progress made since then.

1 Introduction

Automated map generalization is one of the most interesting topics in Geographic Information Systems research. There are many issues on what must be done to visualize or use geographic data at a smaller scale than the source data itself. One is how to organize the data into a multi-scale database to allow for generalization. Another is when and how to detect whether generalization is necessary. A third issue is the design of operators that perform a little piece of the generalization task. Three books and many papers have been written on various issues of cartographic generalization [4, 12, 14]. A survey paper on generalization methods also exists [22].

This paper reviews briefly some of the ideas and concepts developed in the context of automated generalization. The focus is on generalization operators and their design; it is the subarea where computational geometry can have an important contribution. In this paper twelve problems arising in automated cartographic generalization are formulated, with the aim of invoking more research on the topic from the computational geometry community.[1]

[1] The author would appreciate it very much if progress or results on any of the problems are communicated to him too, in one form or another.

Phrasing problems as computational geometry problems requires a complete specification of conditions and criteria. In the GIS literature it is common to say that generalization should be done when imperceptibility, coalescence, or congestion (defined later) occurs. This is done by a conflict detection routine [2]. This paper takes a similar approach but formulates various generalization operators as computational problems where certain restrictions must be met and other conditions are optimized. This leads to several interesting computational geometry problems. But we don't claim that the given formulations are the best possible, in the sense of being appropriate and easy to compute with. For the related problem of line simplification, various geometric conditions and measures have been given before [11, 9].

2 Preliminaries

An operator that performs a piece of the cartographic generalization process makes changes to a detailed representation to obtain a representation that is more suitable at a smaller map scale. Changes are in some sense a lie about the actual situation. Map features may have been eliminated or shown in a simplified shape. Although changes are necessary, they must obey certain constraints. At the same time, the changes serve a generalization objective. This gives a couple of conflicting issues which we formalize first. (A more extensive description of the concepts and ideas in this section is given in [12, 16, 18] and other texts.)

Need of generalization. There are several reasons to perform generalization—apply a generalization operator—to a particular situation on a map. One of them is *imperceptibility*. This occurs when a feature becomes so small on the map at reduced scale that it can hardly be seen, or recognized. Either feature should be removed, if it is not so important, or it should be enlarged, if it is important. Imperceptibility can apply to a feature as a whole, but also to a part of a feature.

Coalescence occurs when two map features seem to touch because their distance is very small on the map at a reduced scale, which makes it hard see them as non-touching features. It can also be that on the map, the features really touch caused by the thickness with which the boundary is drawn. *Self-coalescence* occurs when a polygonal line seems to touch itself.

Another reason for generalization is to reduce *clutter* or *congestion*. If a map shows many features in large detail at a reduced map scale, the map will appear congested with information. The overall impression, which is the purpose of maps with smaller scales, can easily be lost in the detail. Also in specific regions of a map it can be necessary to reduce detail, just to make the reduction in detail roughly the same everwhere on the map.

Means of generalization. The various operators that are used in the generalization process can change a map in the following ways (or perhaps in a combination).

- Selection (or the inverse, elimination).
- Displacement (translation, perhaps in combination with a slight rotation).
- Reshaping. Among which smoothing, simplification, exaggeration, and enhancement of a polygonal line.
- Amalgamation. When two or more polygons close together are merged into a larger polygon.

Figure 1: Example of cartographic generalization with the effect of various operators shown: point conversion for the city, area collapse (partial) for the river, amalgamation of polygons at the top left, typification of the symbols below the river, and selection of roads.

- Typification. When a set of features is displayed by a different set that can be considered representative of the situation. Usually, the new set contains fewer but larger features.
- Exaggeration or enlargement. When a feature is imperceptible or simply too small for its importance, it can be enlarged.
- Point, line, and area conversion. Point conversion is when a small area is symbolized to a point (e.g., a city); line conversion is when a thin region (river, road) is collapsed to a polygonal line or curve; area conversion is when a set of points are displayed as a polygon (a set of oil wells becomes an oil field). Point and line conversion are sometimes called collapse.

Elimination can consist of removing a separate map feature, but also removing a face from a subdivision. In this case the region must be merged to one or more neighboring regions, which are reshaped. This operator is also called dissolution.

Other operators exist, for instance segmentation, aggregation, refinement, and classification. Simplification is considered generization too if the objective is not to reduce the number of data points, but to redefine a shape.

Restrictive conditions. Certain constraints must be satisfied in specific situations of generalization. These constraints may depend on the type of feature. A forest and an island can both be represented as a polygon, but different considerations apply when generalizing. The following conditions can occur:

- Minimum area of a polygon (no imperceptibility).
- Minimum separation between non-adjacent features (no coalescence).
- Minimum width of a polygon (no self-coalescence).

- Maximum number of features in a limited area (no congestion).
- Maximum total length of boundaries in a limited area (no congestion).
- Maintain the topology (e.g., of a subdivision when its faces are generalized).
- Maintain the global shape, size, orientation, position, relative size of features.

The latter class of conditions states that certain aspects must be met more or less, without a firm statement. The other four conditions can be considered as firm: no polygon on the map should be less than $0.5\,\mathsf{mm}^2$, no two features should be closer than $0.1\,\mathsf{mm}^2$ or they will seem to touch, and no polygon should be more narrow than $0.1\,\mathsf{mm}^2$ or the opposite boundaries will seem to touch.

There are more restrictive conditions than were listed above. Some conditions are needed for semantics or consistency. For example, when contour lines are smoothed during generalization, one must make sure than any river crossing the contour lines still flows downhill everywhere. As a second example, if a city is collapsed from an area to a point symbol, then every other city of the same or lesser size should also be collapsed. Restrictive conditions beyond the geometric ones are given, for example, by Ruas and Plazanet [17], Weibel [23].

The type of width or narrowness meant in the preceding text is not the 'largest width' but the 'narrowest passage'. We give a definition of coalescence that more or less captures the notion of narrowness in the next section.

Smallest lie or maximum generalization. Within the limits given by the restrictive conditions, an operator should usually do as little reshaping or displacement as possible. The operator introduces a small lie about the real world situation, and this lie should be no larger than necessary. In some situations, however, it can be desirable to reduce clutter as much as possible within certain limits. The resulting problems are in some sense dual: given a maximum allowed change or lie, perform as much generalization as possible.

Depending on the means of generalization, change can be measured in various ways. For displacement, it is obvious to minimize the displacement distance. For selection, it is natural to try to choose the largest subset under certain constraints. For reshaping, it is best to keep the original size and shape as much as possible. To this end, a similarity measure is needed [1, 7]. For example, when a polygon should be reshaped to meet certain criteria, one could look for the polygon satisfying these criteria which has the minimum area of symmetric difference with the original polygon, or minimum Hausdorff distance. Often is it desirable to maintain the area of a feature, in which case area of symmetric difference is a better measure.

Some of the reasons why generalization is needed suggest different optimization criteria. To reduce congestion as much as possible, selection of a smallest subset that can serve as representative can be the optimization criterion. Or, one could reshape a polygonal line to one with smallest length, and a polygon to one with smallest perimeter.

3 Definitions of imperceptibility, coalescence, and congestion

Three geometric conditions that should be overcome by generalization operators have been identified, but not yet quantified. Imperceptibility can simply be stated as requiring a minimum area of areal map features, and a minimum length of linear map features. Point features are always symbolized, possibly to a point symbol, and the size of the symbol that goes with the map scale prevents points from being imperceptible.

Coalescence is another condition that can occur when generalizing and which must be resolved. Since coalescence is a geometric condition, it requires a geometric definition. For two features like polygons, displayed with a boundary of a certain thickness, the obvious condition giving rise to coalescence is that the boundaries are within a certain very small distance. This distance depends on the thickness of the boundaries and the desired minimum separation between displayed features.

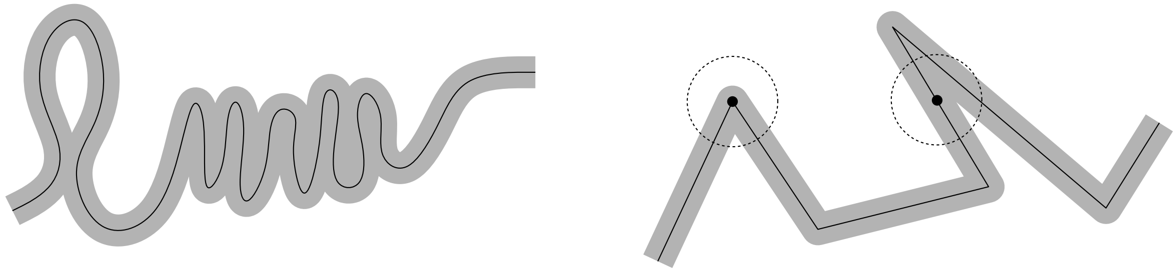

Figure 2: Self-coalescence for a curve.

Coalescence can also occur for a single feature like a polyline or the boundary of a polygon; this is called self-coalescence. This case is not so easy to define appropriately. A polyline should not come so close to itself that it almost touches, nor should it contain such sharp turns that a thicker line appears locally (see Figure 2, left). Similarly, a polygon's boundary can seem to touch itself or contain sharp turns. The idea is to define that two points p, q on a polyline or boundary coalesce if they are closer than some distance δ in the Euclidean sense, but further than some distance γ when measured along the polyline or boundary. Along the boundary of a polygon, distance can be measured clockwise or counterclockwise from p to q. Both distances should be at least γ, which can be chosen to be, say, 3 times the Euclidean distance between p and q. The value of δ will depend on the line thickness too. The requirement rules out sharp turns as well, except when at least one of the segments making the sharp turn is short.

The definition just given may be difficult to work with, because for any point on the boundary of the polygon there can be infinitely many other points on the boundary with which it coalesces. A possibly more workable definition is considering only pairs of points p, q on the boundary for which there exists a midpoint r interior to P such that p, q are the points on the boundary closest to r. The remainder of the definition of coalescence is the same. It follows that r lies on the Voronoi diagram of the edges and vertices of P.

The condition that a polyline can seem thicker than the width with which it is drawn can be stated by measuring locally the area a polyline with a certain width has. For instance, one can require that every circle with radius twice the polyline width may be filled by the polyline for at most, say, 70% of its area. As a variation, one could only consider circles whose center lies on the polyline (see Figure 2, right). This rules out that a polyline can seem much thicker, and also sharp angles of two line segments of at least some length.

We will not formalize the notion of congestion, but simply remark that reducing the number of features and reducing the length of a polyline or curve reduces congestion. A possible definition has been given by Jansen and van Kreveld [8].

The definitions given above may not be the best ones imaginable. For one thing, a boundary of a polygon can be arbitrarily long in a very small region, for instance some fractals have an infintely long distance over the curve between any two points. Still, such a curve isn't necessarily infinitely congested, nor need it cause self-coalescence. This suggests that it may be better to replace length of a curve between two points by area of the Minkowski sum of a small disk and the curve between those points.

4 Twelve problems in generalization

In the next twelve subsections, a problem from cartographic generalization is discussed briefly. Sometimes, the problem is formulated as a completely specified problem; other problems still need to have part of the problem statement filled in. The problems are examples of general generalization problems, with choices made on the conditions and optimizations. So variations of the listed problems are also of interest. The author of this text has thought about solutions for a few of the problems listed in this section, but not about all of them. So some problems may turn out te be simple, others extremely difficult, or even uninteresting.

4.1 Reshape polygon to counter self-coalescence

The first generalization problem states what to do with a polygon when self-coalescence occurs. This immediately gives rise to the problem of detecting whether self-coalescence occurs, using one of the definitions of self-coalescence given before. When it occurs, the polygon can be reshaped slightly so that self-coalescence no longer occurs. We would like to minimize the area of symmetric difference.

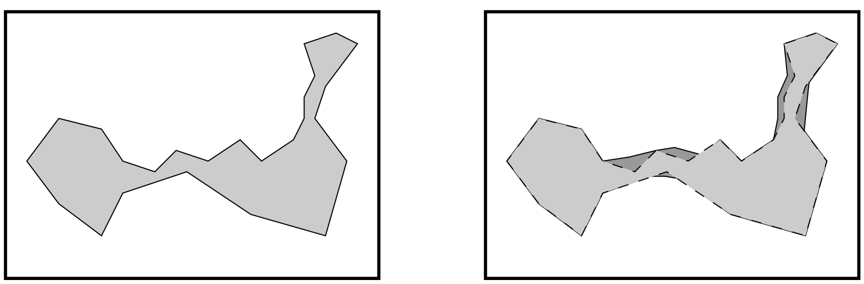

Figure 3: Reshaping one polygon to garantee no self-coalescence.

One can also reshape a polygon to one with minimum perimeter, while guaranteeing that the (symmetric) Hausdorff distance between the old and new boundaries is at most some small value ϵ. Small perimeter is used to deal with congestion, but need not resolve self-coalescence. Requiring that the resulting reshaped polygon has no self-coalescence is a more difficult version of this problem.

4.2 Reshape two polygons to counter coalescence

To guarantee a minimum separation (no coalescence) between map features, one can change the shape of those map features. For instance, two polygons may represent two regions with a different type of land use, and they may be some distance apart. To visualize that two polygons don't touch when reducing the scale of the map, a minimum separation is necessary.

This may be obtained by changing the shapes of the polygons. We may want to minimize the area that is taken from the polygons to guarantee the minimum separation.

When only one polygon should change its shape, it is straightforward how to compute its new shape. Simply carve all parts that intersect a thickened version of the other polygon,

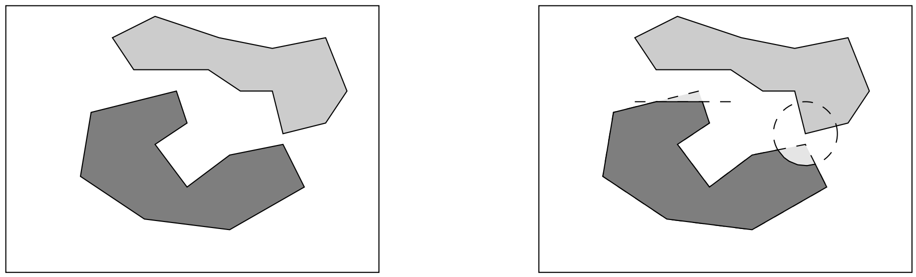

Figure 4: Reshaping one polygon to garantee a separation.

obtained by applying the Minkowski sum of that other polygon with a disk. When two polygons may change shape and the total area taken from the two is to be minimized, the problem is not so simple.

A practical solution is to compute the bisector of the two polygons, compute its Minkowski sum with a disk, and remove all parts of the polygons intersecting this Minkowski sum. The amount of area taken from the polygons may be much more than in an optimal solution, though. It remains to find an optimal solution or a constant factor approximation of it.

Ultimately, the problem can be stated for a set of polygons, some sharing boundaries and some disjoint. Reshape all to guarantee no coalescence and self-coalescence anywhere, while maintaining the topology of the subdivision and minimizing the total area of symmetric difference.

4.3 Displace polygon to counter coalescence

The situation of the problem discussed in this subsection is the same as the previous one. Two polygons are given and a minimum separation must be guaranteed. Another way to achieve this, other than reshaping, is to displace one of the polygons. The distance over which the polygon is translated to guarantee the separation should be minimized.

The obvious way to tackle the problem is to reduce the polygon P to be translated to a point and the other one, Q, to the Minkowski sum $P \oplus Q$ of the two polygons. Then the problem becomes a motion planning problem for a point p. To guarantee a minimum separation, we enlarge $P \oplus Q$ by taking the Minkowski sum with a disk D of the appropriate radius, yielding a subset of the plane $R = P \oplus Q \oplus D$. The point p lies outside $P \oplus Q$ but outside R. The motion planning problem is to find the nearest point on the boundary of R to which p can be translated. There are no obstacles in this motion planning problem, but the target isn't a single point.

The solution just outlined may be expensive: the Minkowski sum of two polygons with n and m vertices, respectively, may have complexity $\Theta((nm)^2)$ in the worst case. Does a more efficient solution exist?

A version of the problem where polygon P may not intersect polygon Q during its translation is also interesting in the application. In Figure 5, the dark polygon P will be translated

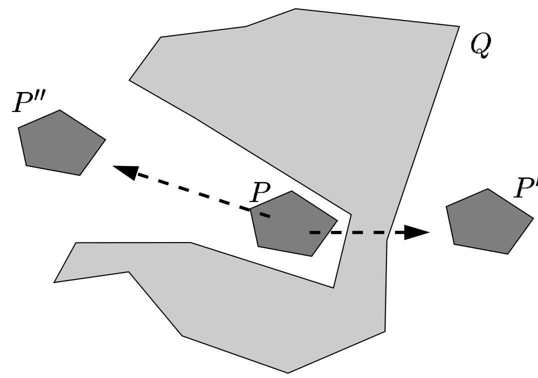

Figure 5: Candidate positions of polygon P after translation.

to the right position P' in the solution of the problem just sketched. However, it may be important to keep the intuition that polygon P lies to the left of Q. So the left position P'' of the dark polygon may be better, even though it is further from the original position.

4.4 Select subset of points to counter congestion

A collection of symbols close together may cause a lot of clutter on a map. This applies to point symbols and other symbols. Sometimes it is important for the message of a map that a region contains several castles, but it is not necessary on smaller map scales to show a castle symbol for every castle that occurs, when there are many in a small region. A representative

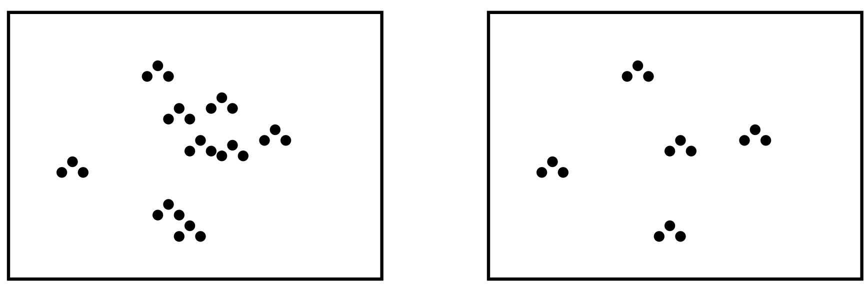

Figure 6: Selecting a subset of symbols to overcome clutter.

subset can be used to convey this message. Figure 6 makes clear that the three dots commonly used as symbol for a ruin can make a part of a map appear cluttered.

A solution to assure that no clutter of a symbol type occurs is to guarantee a minimum distance between two symbols of the same type. When two such symbols are two close, one of them must be removed. While guaranteeing a minimum separation, one may want to keep the largest subset of symbols with this restriction. Assuming that symbols are points, the problem can then be seen as maximum nonintersecting subset in a set of unit size disks (or maximum independent set in a unit disk graph).

It may also be the goal to reduce clutter as much as possible, but while maintaining the property that every eliminated symbol has a representative within a certain distance from it.

This distance should obviously be chosen larger than the minimum separation distance. The problem then becomes a cover type of problem.

4.5 Amalgamate polygons to counter coalescence and congestion

Amalgamation is merging two or more polygons into one by adding the area in between [3]. This can be done to cope with polygons that give rise to coalescence over a longer section of their boundaries. The first problem of this type is: Given two polygons P and Q, amalgamate them into one polygon with no self-coalescence while adding the minimum possible area.

Given two polygons P and Q, amalgamate them into one polygon with no self-coalescence and smallest possible perimeter, under the condition that no point further than some distance ϵ from either polygon be added.

More difficult is to solve the problems above and also guarantee that the complement of the resulting polygon has no self-coalescece. Another problem is to start out with a set of polygons and amalgamate them into a smaller set of polygons under certain restrictions.

4.6 Partial area collapse to counter self-coalescence

On large scale maps, rivers and roads are displayed as polygonal regions, and the width of these features can be shown at the right scale. On small scale maps this is not possible any more, and rivers must be collapsed to curves or polygonal lines. At intermediate scales, rivers can be partly areal, partly linear. The areal sections of the river show where the river is wide or even forms a lake.

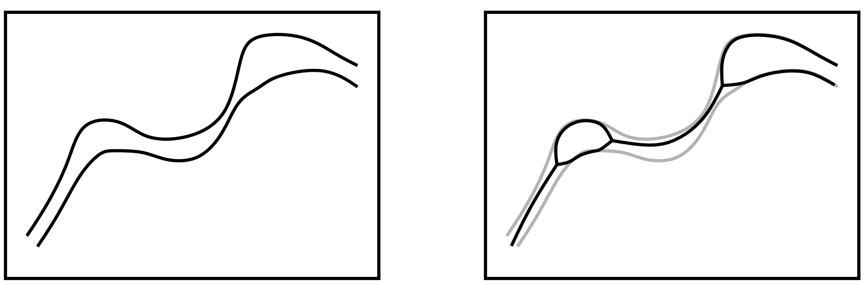

Figure 7: Partial area collapse.

Given a target map scale, when a river is narrow enough, it can be collapsed to a curve or polyline, but if it's wider than some value, it should not be collapsed. It is a good idea to take two different thresholds w_1, w_2: below some width w_1 the river must be collapsed, between widths w_1 and w_2 one has the choice whether to collapse the river or not, and above width w_2 the river may not be collapsed. Under these restrictive conditions, the problem is to compute the partially collapsed feature such that the number alternations in collapse and not collapse is minimized. When this is optimized, one may still choose to collapse as large sections as possible, or as small sections.

4.7 Area conversion to counter congestion

The generalization operator area conversion can be applied to a set of points close together, giving a polygon [5]. For instance, a set of oil wells can be area-converted to an oil field. The operator resembles amalgamation of polygons into one new polygon, and α-hulls or shapes could be used [6, 13].

Here we pose the problem as a cartographic optimization problem with restrictive conditions. Given a set S of points in the plane, determine a set of polygons such that every point lies in some polygon, there is no coalescence or self-coalescence, and the total area of all polygons is minimized. One could also add the condition that no point further than some distance ϵ from all points in S be in any polygon, and minimize the total perimeter of the polygons.

4.8 Typification in sets of polygons to counter congestion

Typification is one of the more difficult to formalize operators in cartographic situation. The idea is to replace a large number of small polygons by a much smaller number of larger polygons, but which have a representative shape and orientation. To formulate the problem well, the notion of when a set of polygons represents another set well must be formalized. As in most other problems, it is desirable that the new polygons together have roughly the same area as the input set. Furthermore, the new polygons may not be imperceptible not have coalescence.

A possibility of typifying the shape and orientation of a set S of polygons by one new polygon P is to scale all polygons in S (such that the average area is the same as the area of P), then translating P to minimize the area of symmetric difference of P and each scaled copy in S, and adding these. This gives a way to measure the resemblance of P with all polygons in S, and one can look for the polygon P which minimizes this sum of areas of symmetric difference.

Different similarity measures give rise to more problems of this type [1, 7]. Also, a formulation of a representative set of polygons for an input set need be found.

4.9 Dissolution in a subdivision to counter imperceptibility and congestion

When a face in a subdivision becomes too small (imperceptible), it can either be enlarged or eliminated (when there are several small faces of the same type, they could also be amalgamated). If a face is eliminated, its area is given to adjacent faces of the subdivision. Because the eliminated face is usually small, the adjacent faces won't change much in size or shape. Nevertheless, it may be necessary to subdivide the eliminated face in a particular way among its neighbors. It could be that the face is not that small, but simply not so important, or that several faces of the subdivision are eliminated and one wants to avoid that they disturb the relative sizes of the remaining faces too much. The problem occurs when generalizing categorial subdivisions, for instance of soil types.

In Figure 8, most of the small dark faces were dissoluted in the white face, but the two small dark faces adjacent to the light grey face were added to that face and not the white face to maintain the relative areas of white and light grey more or less.

The problem of dissoluting one face can be stated as how to divide its area among its neighbors while maintaining their relative sizes, and at the same time, using short boundaries. When the subdivision is categorial, the objective is to maintain the relative sizes of the classes as much as possible.

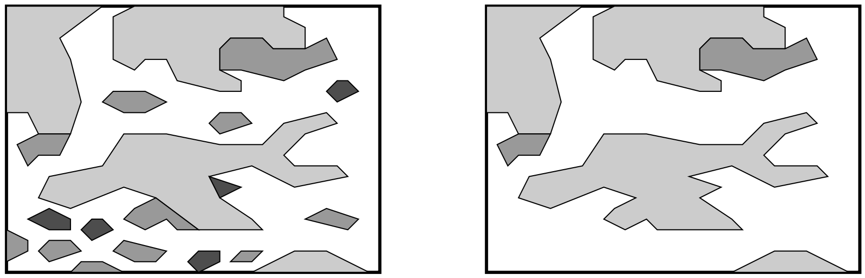

Figure 8: Dissolution of small faces in a subdivision.

4.10 Exaggeration amidst other features

Exaggeration is always performed to deal with imperceptibility of a feature or an important part of it. Here we assume that a polygon is too small to be displayed at the target scale, but it is important enough not to be eliminated. So it has to be exaggerated.

The best way to enlarge a polygon while maintaining its shape obviously is scaling it. However, there can be other polygons in the neighborhood with it which would coalesce. The problem of exaggerating a polygon P then becomes computing a new polygon Q with a given target area, whcih doesn't coalesce with any other polygon, and whose area of symmetric difference with some scaled copy of P is minimum. The center of scaling should lie inside P.

4.11 Exaggeration in a subdivision

When a subdivision has small faces but these may not be eliminated, they have to be enlarged. For example, when a choropleth map of Europe shows the countries with a color scheme that shows unemployment rate, also the small countries like Liechtenstein and Monaco must be visible. On a scale 1:40,000,000 (about 10×10 cm for Europe), these countries must be enlarged to ensure a minimum size. This should disturb the shape and size of neighboring countries as little as possible, which may be difficult if these neighboring countries were fairly small themselves. A restrictive condition may be to maintain the topological structure of the subdivision.

4.12 Typification of a polyline to counter self-coalescence

When generalizing a winding road, curves tend to become so small that they cannot be visualized. A road with many hairpin turns will appear as a thick line rather than a road over a steep mountain. It is possible to apply line generalization or simplification and throw out all turns, but then the road will look like a normal road and not a winding road. To preserve the character, it is best to reduce the number of turns in the road, but exaggerate the remaining ones. The most interesting approach of the problem is by Plazanet [15].

A possible problem statement is to compute a curve with an upper bound on its curvature, with the same angular change to length ratio as the input curve, and with Hausdorff distance as small as possible to the original curve. An additional constraint should be to avoid self-coalescence, for example by requiring the complement of the curve to be sufficiently wide.

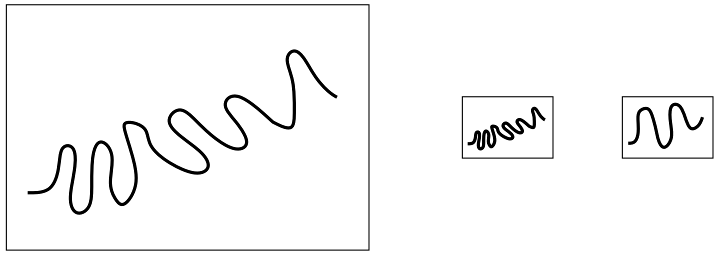

Figure 9: Generalizing a road with many curves by keeping a few curves and exaggating them.

4.13 And more ...

There are several other problems of interest. The operator segmentation, which splits a polygon into two at a thin place, can also be stated as a problem. Sometimes, reshaping a polygon as proposed in Subsection 4.1 is not possible or would add too much area. In that case segmentation can be better.

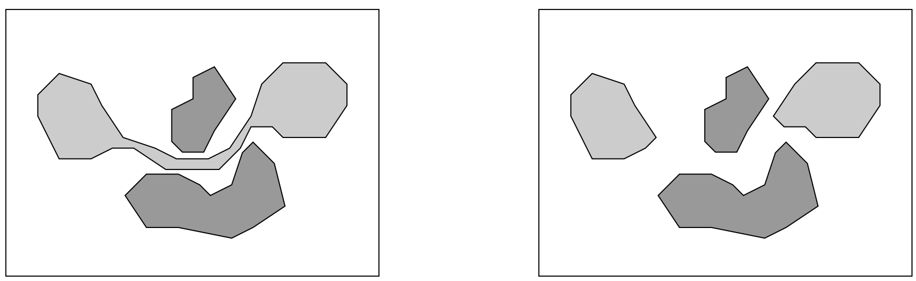

Figure 10: Segmentation of a polygon.

In the formulations given in this paper we specified which operator to use in a particular situation. It is part of fully automated generalization to select the most appropriate operator as well. Certain problems, like the one on dissolution of a face in a catergorial subdivision, should be solved by a combination of amalgamation, dissolution, segmentation, and reshaping.

Several problems in generalization include features that have a priority as well, not just geometric information. For instance, when generalizing road network maps, one should give preference to keeping the major roads (but it is not the case that road selection by priority solves the problem) [20, 19]. In road network generalization, junction collapse is also an operator [10].

In other problems, certain high level structures of the data need be maintained. For example, when generalizing digital elevation models, it is important to keep the ridges and river courses roughly the same [21]. Similarly, when selecting a subset of points from a dense set that has some pattern, this pattern should be maintained.

5 Concluding remarks

The list of problems given in this paper is by no means exhaustive. The development of automated generalization will benefit from their solution, but will not solve it. The interplay of generalization operators, the order in which to perform them, the inclusion of shape descriptors to maintain global shape, the more explicit incorporation of meaning (semantics) to the map objects, consistency of generalization throughout the map, generalization of complex structures like elevation models and networks, these are all further issues in generalization, all of which have been addressed to some extent in the recent GIS literature.

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